What aggregation rules can be classified as logical concepts?

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XXX EUROPEAN WORKSHOP ON ECONOMIC THEORY
Special session in memory of Jerzy Łoś
July 7 – 9, 2022

Observations and motivation

Classic results of Computational Social Choice are the impossibility and possibility theorems.

Impossibility theorems

In 1785, le marquis de Condorcet observed that majority rule does not preserve a set of rational preferences on a set (of alternative) A of cardinality $|A| \geqslant 3$.

In 1951, Kenneth Arrow proved that there are no non-dictatorship local aggregation rules that preserve a set of rational preferences on a finite set A of cardinality $|A| \geqslant 3$.

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In 2005, Saharon Shelah showed that, under some additional conditions, Arrow's impossibility principle applies to any symmetric set of preferences.

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Shelah's theorem on the Arrow property and its refinement

Notation and definitions.

- A – a finite non-empty set (of alternatives).
- $\mathfrak{C}_r(A)$ – the set of all choice functions $\mathfrak{c} : [A]^r \to A$ where r is a natural number and $[A]^r$ is a set of all r-element subsets of A. Functions $\mathfrak{c} \in \mathfrak{C}_r(A)$ represent individual preferences.
- A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is *symmetric* if

$$\mathfrak{c} \in \mathfrak{D} \Rightarrow \mathfrak{c}_\sigma \in \mathfrak{D}$$

  for all permutations σ of A where $\mathfrak{c}_\sigma$ is a choice function defined by

$$\mathfrak{c}_\sigma(p) = \sigma^{-1}\mathfrak{c}(\sigma p)$$

  for all $p \in [A]^r$.
- Any function $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is called an *aggregation rule*.
- An aggregation rule $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ *preserves* a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ if

$$f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n) \in \mathfrak{D}$$

  for all functions $\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n \in \mathfrak{D}$.
- An aggregation rule $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is called a *dictatorship* rule if it is a projection.

- An aggregation rule $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is *local* if, for all $\mathfrak{c}_1$, $\mathfrak{c}_2$, ..., $\mathfrak{c}_n$, $\mathfrak{d}_1$, $\mathfrak{d}_2$, ..., $\mathfrak{d}_n \in \mathfrak{C}_r(A)$ and $p \in [A]^r$,

1. $(\mathfrak{c}_1(p) = \mathfrak{d}_1(p) \wedge \ldots \wedge \mathfrak{c}_n(p) = \mathfrak{d}_n(p)) \Rightarrow f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(p) = f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n)(p)$ (f acts pointwisely),
2. $f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n)(p) \in \{\mathfrak{d}_1(p), \mathfrak{d}_2(p), \ldots, \mathfrak{d}_n(p)\}$ (f is pointwisely conservative).

- A set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ has *the Arrow property* if every local aggregation rule that preserves it is a dictatorship rule.

Theorem (S. Shelah, 2005)

If $7 \leqslant r \leqslant |A| - 7$, than any non-empty proper subset $\mathfrak{D}$ of $\mathfrak{C}_r(A)$ has the Arrow property.

A complete classification of symmetric sets $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ with the Arrow property (for all finite r) was obtained in 2014 – 2016.

- Polyakov, N.L., Shamolin, M.V.: On a generalization of Arrow's impossibility theorem. Dokl. Math., 89, 290–292, 2014 (only announced).
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Possibility theorems state that some set $\mathfrak{D}$ of preferences is preserved by some aggregation rule f, or at least that the application of f to preferences from $\mathfrak{D}$ does not lead beyond some set $\mathfrak{C}$ (for example, the set of rational preferences).

Apparently, the first informative example (*single peaked domains*) belongs to D. Black.

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Shelah's theorem (and its refinement) does not allow us to hope that, in reasonable cases, the possibility theorems will be true for any non-trivial set of preferences that is closed under permutations of the set of alternatives. However, the following fact holds:

majority rule μ *(as well as many other rules) has some non-trivial invariant preference sets* $\mathfrak{D}$ *such that any preference set* $\mathfrak{D}_\sigma$ *obtained from* $\mathfrak{D}$ *by a permutation* σ *of alternatives is also preserved by* μ

(for the majority rule, this condition is satisfied for all invariant sets $\mathfrak{D}$).

We believe that this is an important property of aggregation rules. Indeed, if an invariant $\mathfrak{D}$ of the aggregation rule f can be determined by a set-theoretic formula without constants from a set of alternatives A, then all sets $\mathfrak{D}_\sigma$ are also invariants of f.

In addition, we believe that in this case the aggregation on the set D by means of the rule f can be understood as a certain logical procedure. As an argument, we will refer to the statement of Alfred Tarski on the nature of logical concepts:

"*...a notion (individual, set, function, etc) based on a fundamental universe of discourse is said to be logical if and only if it is carried onto itself by each one-one function whose domain and range both coincide with the entire universe of discourse*".

- Tarski, A. What are logical notions? History and Philosophy of Logic, 7, 143–154, 1986.

The above motivates us to the following task: to describe all local (in some natural sense) aggregation rules that have a non-trivial class $\mathcal{D}$ of invariant preference sets $\mathfrak{D}$ that is closed under permutations.

We argue that this problem is solvable in a very general situation, but leads to an overly broad classification. However, if we restrict ourselves to the case of preference aggregation from $\mathfrak{C}_2(A)$ (of a choice function on $[A]^2$), we get a beautiful and, we hope, useful result. Note that this case includes many classical problems of Computational Social Choice, including the problem of aggregation of rational preferences.

In this work, we apply the clone approach proposed by S. Shelah. The essence of this approach will also be briefly outlined.

General settings

Let

- C – a non-empty set (of *conditions*),
- D – a non-empty set (of *decisions*), $|D| \leqslant |C|$,
- $DM \subseteq D^C$ – a non-empty set (of *individual decision making functions*),
- $*$ – a fixed embedding $S_D \to S_C$ where S_X is a permutation group of a set X.

Examples

In the most typical situation,

- C is a set of subsets of D perhaps enriched by some additional structure and closed under isomorphisms,
- DM is a set of all choice functions defined on C (i.e., functions $\mathfrak{c} : C \to D$ satisfying $\mathfrak{c}(x) \in x$ for all $x \in C$),
- for any $\sigma \in S_D$ and $c \in C$, $\sigma^*(c)$ is the isomorphic image of c under isomorphism $\sigma\restriction_c$ (we write σ^* instead of $*(\sigma)$).

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Definitions

- For any function $\mathfrak{d} \in DM$ and permutation $\sigma \in S_D$ the function $\mathfrak{d}_\sigma : C \to D$ is defined by
$$\mathfrak{d}_\sigma(c) = \sigma^{-1}\mathfrak{d}(\sigma^*(c))$$
for all $c \in C$.

- For any set $\mathfrak{D} \subseteq DM$ and permutation $\sigma \in S_D$ let
$$\mathfrak{D}_\sigma = \{\mathfrak{d}_\sigma : \mathfrak{d} \in \mathfrak{D}\}.$$

- A set $\mathfrak{D} \subseteq DM$ is *symmetric* if
$$\mathfrak{D}_\sigma = \mathfrak{D}$$
for any permutation $\sigma \in S_D$.

- A set $\mathcal{D} \subseteq \mathscr{P}(DM)$ is *symmetric* if
$$\mathfrak{D} \in \mathcal{D} \Rightarrow \mathfrak{D}_\sigma \in \mathcal{D}$$
for any permutation $\sigma \in S_D$ and set $\mathfrak{D} \subseteq DM$.

Examples

Let C be the set $[D]^r$ of all r-element subsets of D, and $DM = \mathfrak{C}_r(D)$.

The following sets $\mathfrak{D} \subseteq DM$ are symmetric:

- $\varnothing$,
- DM,
- the set $\mathfrak{R}_r$ of all *rational* choice functions $\mathfrak{d} : [D]^r \to D$, i.e. of all choice functions $\mathfrak{d}_\succ$ satisfying

$$\mathfrak{d}_\succ(x) = \max_\succ x$$

  where $\succ$ is an arbitrary linear order on D,
- etc...

The following sets $\mathcal{D} \subseteq \mathscr{P}(DM)$ are symmetric:

- any singleton $\{\mathfrak{D}\}$ for an arbitrary symmetric set $\mathfrak{D} \subseteq DM$,
- the set of all singletons $\{\mathfrak{d}\}$, $\mathfrak{d} \in DM$,
- the set $\{\{\mathfrak{d}_\succ : \succ \in SP_>\} : > \text{ is a linear order on } D\}$ where $SP_>$ is a single peaked domain generated by $>$,
- etc...

Definitions

For any $n \in \mathbb{N}$, any function $f : DM^n \to DM$ is called an *aggregation rule*.

An aggregation rule $f : DM^n \to DM$ *preserves* a set $\mathfrak{D} \subseteq DM$ and a set $\mathfrak{D} \subseteq DM$ is *preserved* by aggregation rule $f : DM^n \to DM$ (or $\mathfrak{D}$ is an *invariant* of f) if

$$f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n) \in \mathfrak{D}$$

for all functions $\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n \in \mathfrak{D}$.

An aggregation rule $f : DM^n \to DM$ is called a *dictatorship* rule if it is a projection, i.e. if there is a number i, $1 \leqslant i \leqslant n$ such that

$$f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n) = \mathfrak{d}_i.$$

Obviously, a dictatorship rule preserves any set $\mathfrak{D} \subseteq DM$.

Definition

An aggregation rule $f : DM^n \to DM$ is *local* (or satisfies the *Arrow conditions*) if, for all $\mathfrak{c}_1$, $\mathfrak{c}_2$, …, $\mathfrak{c}_n$, $\mathfrak{d}_1$, $\mathfrak{d}_2$, …, $\mathfrak{d}_n \in DM$ and $a \in C$,

1. $(\mathfrak{c}_1(a) = \mathfrak{d}_1(a) \wedge \ldots \wedge \mathfrak{c}_n(a) = \mathfrak{d}_n(a)) \Rightarrow f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(a) = f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n)(a)$ (f acts pointwisely),
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An aggregation rule $f : DM^n \to DM$ is *simple* if, for all $\mathfrak{d}_1$, $\mathfrak{d}_2$, …, $\mathfrak{d}_n \in DM$ and $a, b \in C$,

3. $(\mathfrak{d}_1(a) = \mathfrak{d}_1(b) \wedge \ldots \wedge \mathfrak{d}_n(a) = \mathfrak{d}_n(b)) \Rightarrow f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n)(a) = f(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n)(b)$.

Examples

- Any dictatorship rule is local and simple.
- If an aggregation rule f is local and $|\{\mathfrak{d}(a) : \mathfrak{d} \in DM\}| \leqslant 2$ for any $a \in C$ (the case of binary choice), then f is simple.
- Majority rule is local and simple.

Definition

An aggregation rule $f : DM^n \to DM$ is *local* (or satisfies the *Arrow conditions*) if, for all $\mathfrak{c}_1$, $\mathfrak{c}_2$, ..., $\mathfrak{c}_n$, $\mathfrak{d}_1$, $\mathfrak{d}_2$, ..., $\mathfrak{d}_n \in DM$ and $a \in C$,

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An aggregation rule $f : DM^n \to DM$ is *represented* by a function $g : X \to D$ where $X \subseteq D^n$ if, for all $\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_n \in DM$ and $a \in C$,

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Definition

Any function $f : X \to D$ where $X \subseteq D^n$ is called *conservative* if

$$f(x_1, x_2, \ldots, x_n) \in \{x_1, x_2, \ldots, x_n\}$$

for all $x_1, x_2, \ldots, x_n \in X$.

Claim

Any local and simple aggregation rule $f : DM^n \to DM$ is represented by a conservative function $g : D^n \to D$.

If $|\{\mathfrak{d}(a) : \mathfrak{d} \in DM\}| = r$ and DM is symmetric, then any local and simple aggregation rule $f : DM^n \to DM$ is represented by the unique conservative function $\widehat{f} : D^n_{\leqslant r} \to D$ where $D^n_{\leqslant r} = \{\boldsymbol{x} \in D^n : |\operatorname{ran} \boldsymbol{x}| \leqslant r\}$.

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If $|\{\mathfrak{d}(a) : \mathfrak{d} \in DM\}| = r$ and DM is symmetric, then any local and simple aggregation rule $f : DM^n \to DM$ is represented by the unique conservative function $\widehat{f} : D^n_{\leqslant r} \to D$ where $D^n_{\leqslant r} = \{\boldsymbol{x} \in D^n : |\mathrm{ran}\, \boldsymbol{x}| \leqslant r\}$.

Clonal approach

Definition

A *clone* (on a set X) $\mathcal{F}$ is a set of functions $f \in \bigcup_{n<\omega} X^{X^n}$ closed under superposition and containing all projections.

A clone $\mathcal{F}$ is *conservative* if it contains only conservative functions.

A clone $\mathcal{F}$ on a set X is *symmetric* if for any $n \in \mathbb{N}$, $f \in \mathcal{F}$, and $\sigma \in S_X$ it contains the function f_σ defined by

$$f_\sigma(x_1, x_2, \ldots, x_n) = \sigma^{-1} f(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n)).$$

A clone $\mathcal{F}$ is *generated* by a function g (by a set of functions $\mathcal{G}$) if it is the minimal (under inclusion) clone containing g (respectively, $\mathcal{G}$).

Clones and closed functional classes on finite sets have been carefully studied since the work of E. Post, who constructed a complete classification of closed classes of Boolean functions.

- Post E.: Two-valued iterative systems of mathematical logic. Annal of Math. studies 5. Princeton University Press, Princeton (1942).

For any aggregation rule f, the set of all sets $\mathfrak{D} \subseteq DM$ that are invariant for f is denoted by $\operatorname{Inv} f$. For any set $\mathcal{F}$ of aggregation rules we denote $\operatorname{Inv} \mathcal{F} = \bigcap_{f \in \mathcal{F}} \operatorname{Inv} f$.

Symmetrically, $\operatorname{Pres} \mathfrak{D}$ is a set of all aggregation rules f that preserve a set $\mathfrak{D} \subseteq DM$, and $\operatorname{Pres} \mathcal{D} = \bigcap_{\mathfrak{D} \in \mathcal{D}} \operatorname{Pres} \mathfrak{D}$ for any $\mathcal{D} \subseteq \mathscr{P}(DM)$.

The set of all local and simple aggregation rules is denoted by $\mathcal{LS}$.

Claim

1. *For any set $\mathcal{D} \subseteq \mathscr{P}(DM)$ the sets $\operatorname{Pres} \mathcal{D}$ and $\operatorname{Pres} \mathcal{D} \cap \mathcal{LS}$ are clones on DM.*
2. *For any set $\mathcal{D} \subseteq \mathscr{P}(DM)$ the set $\operatorname{Pres}_0 \mathcal{D}$ of all conservative functions g what represent functions $f \in \operatorname{Pres} \mathcal{D} \cap \mathcal{LS}$ is a conservative clone on D.*
3. *If a set $\mathcal{D}$ is symmetric then the clone $\operatorname{Pres}_0 \mathcal{D}$ is symmetric.*

Remark

The pair $(\operatorname{Inv}, \operatorname{Pres})$ is an antitone Galois connection between Boolean lattices $\mathscr{P}(AR)$ and $\mathscr{P}(\mathscr{P}(DM))$.

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Definition

A set $\mathfrak{D} \subseteq DM$ is *trivial* if either it is empty or there is a set $X \subseteq C$ and a function $\mathfrak{c} : X \to D$ such that

$$\mathfrak{D} = \{\mathfrak{d} \in DM : d \restriction_X = \mathfrak{c}\}.$$

A set $\mathcal{D} \subseteq \mathscr{P}(DM)$ is *trivial* if it consists only of trivial sets $\mathfrak{D} \subseteq DM$.

Examples

The following sets $\mathcal{D} \subseteq \mathscr{P}(DM)$ are trivial:

1. $\varnothing$,
2. $\{DM\}$,
3. $\{\{\mathfrak{d}\} : \mathfrak{d} \in DM\}$,
4. etc...

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Each trivial set $\mathfrak{D} \subseteq DM$ is preserved by any local and simple aggregation rule.

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Let $\mathcal{F}$ be a symmetric conservative clone on a set D, and let there is a conservative function $h \in \bigcup_{n<\omega} D^{D^n}$ that does not belong to $\mathcal{F}$.

Then there is a set C and a set $DM \subseteq D^C$ such that for any $n < \omega$ each n-ary function $g \in \mathcal{F}$ represents some aggregation rule $f : DM^n \to DM$ with a non-trivial symmetric invariant set $\mathfrak{D} \subseteq DM$.

Now we can return to the question: what local and simple non-dictatorship aggregation rules have non-trivial symmetric set of invariants $\mathcal{D} \subseteq \mathscr{P}(DM)$? Neglecting some details, for an answer it is enough to classify all symmetric conservative clones on a set D.

Surprisingly, this problem is solvable for finite sets D. However, the full classification is rather complex. For example, on a four-element set, there are 42 symmetrical conservative clones.

- Polyakov N. L. Galois correspondences for classes of discrete functions and their application to mathematical problems of Social Choice Theory (PhD thesis), 2016.
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Locally Boolean case

Definition

A set $DM \subseteq D^C$ is *locally Boolean* if $|\{\mathfrak{d}(a) : \mathfrak{d} \in DM\}| = 2$ for any $a \in C$.

For example, the set $\mathfrak{C}_2(D)$ of all choice functions $\mathfrak{d} : [D]^2 \to D$ is locally Boolean. In order to obtain other examples, one can replace $[D]^2$ by a set of multisets X with two-element supports $\operatorname{Supp} X \subseteq D$, or by a set $D^n_{\leqslant 2}$ (accordingly changing the concept of the choice function).

Definition

A set $\mathcal{F}$ that consists of conservative functions f defined on a set $D^n_{\leqslant 2}$ $(1 \leqslant n < \omega)$, contains all projection (on $D^n_{\leqslant 2}$) and is closed under superposition is called a 2-clone (the concept of superposition is understood in the natural sense).

In order to find all local and simple aggregation rules with non-trivial symmetric invariants on a locally Boolean set $DM \subseteq D^C$, it suffices to classify all symmetric 2-clones on D. It turns out that if $5 \leqslant |D| < \omega$, there are only 10 such 2-clones.

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Let us list all Boolean clones $\mathcal{F}$, which

1. consist of functions f that preserve **0** and **1**, i.e.

$$f(0,0,\dots,0)=0 \text{ and } f(1,1,1,\dots,1)=1,$$

2. are closed with respect to duality, i.e.

$$f(x_1,x_2,\dots,x_n)\in\mathcal{F}\Rightarrow\overline{f(\overline{x}_1,\overline{x}_2,\dots,\overline{x}_n)}\in\mathcal{F}.$$

Post's classification theorem yields exactly six such clones: O_1, D_1, D_2, L_4, A_4, C_4 (in the Post's notation). These clones and the Boolean functions that generate them are listed in the table:

Clone	Generating functions	Clone	Generating functions
O_1	x	D_1	$\overline{x}y\vee\overline{y}z\vee yz$ or $xy\vee yz\vee xz$, $x\oplus y\oplus z$
D_2	$xy\vee yz\vee xz$	L_4	$x\oplus y\oplus z$
A_4	$xy, x\vee y$	C_4	$x\vee y\overline{z}$

Definition

Let $|D| \geqslant 2$.

1. For any clone $\mathcal{F} \in \{O_1, D_1, D_2, L_4, A_4, C_4\}$, we say that a 2-clone $\mathcal{G}$ on D is the *free extension* of $\mathcal{F}$ if
   1. For any set $X \in [D]^2$ the set $\mathcal{G}_X = \bigcup_{1 \leqslant n < \omega} \{f|_{X^n} : f \in \mathcal{G} \cap D^{D_2^n}\}$ is a clone that is naturally isomorphic to $\mathcal{F}$,
   2. For all $n < \omega$ and $f : D_2^n \to D$, $f \in \mathcal{G}$ iff $f \restriction_{X^n} \in \mathcal{G}_X$ for all $X \in [D]^2$.

   The free extension of $\mathcal{F}$ is denoted by $\mathcal{F}^{\uparrow}(D)$.
2. For any clone $\mathcal{F} \in \{O_1, D_1, D_2, L_4\}$, we say that a 2-clone $\mathcal{G}$ on D is the *dependent extension* of $\mathcal{F}$ if
   1. For any set $X \in [D]^2$ the set $\mathcal{G}_X = \bigcup_{1 \leqslant n < \omega} \{f \restriction_{X^n} : f \in \mathcal{G} \cap D^{D_2^n}\}$ is a clone that is naturally isomorphic to $\mathcal{F}$,
   2. For all $n < \omega$, n-ary function $f \in \mathcal{G}$, sets $X, Y \in [D]^2$, and sequence $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in X^n$

   $$f(x_1, x_2, \ldots, x_n) = \sigma^{-1} f(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n))$$

   for some bijective map $\sigma : X \to Y$.

   The dependent extension of $\mathcal{F}$ is denoted by $\mathcal{F}^{\Uparrow}(D)$.

Theorem

Let $5 \leqslant |D| < \omega$, and let $\mathcal{G}$ be a symmetric 2-clone on D. Then one of two following conditions holds:

1. *$\mathcal{G}$ is the free extension of one of clones O_1, D_1, D_2, L_4, A_4, C_4,*
2. *$\mathcal{G}$ is the dependent extension of one of clones O_1, D_1, D_2, L_4.*

Remark

- The free extension of the clone C_4 contains all conservative functions on D. If the function $g \in C_4^{\uparrow}(D)$ does not belong to any other of the listed clones, then the corresponding aggregation rule has no non-trivial symmetric sets of invariants.
- The depend extension of the clone O_1 contains only projections (what represent the dictatorship rules). Obviously, any set $\mathfrak{D} \in DM$ is preserved by any dictatorship rule.

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The most beautiful result arises if we restrict ourselves to the case when $DM = \mathfrak{C}_2(D)$.

Let λ and μ be the conservative functions defined on $D^3_{\leqslant 2}$ by the identities:

$$\lambda(x,x,y) = \lambda(x,y,x) = \lambda(y,x,x) = y \text{ and } \mu(x,x,y) = \mu(x,y,x) = \mu(y,x,x) = x.$$

For simplicity of formulation, let us agree to identify local aggregation rules and the conservative functions that represent them.

Theorem

Let $5 \leqslant |D| < \omega$. Let $\mathcal{D}$ be a nontrivial symmetric subset of $\mathscr{P}(\mathfrak{C}_2(D))$, and let $\operatorname{Pres}\mathcal{D}$ contain at least one non-dictatorship local aggregation rule. Then one of three following alternatives holds:

1. *any set $\mathfrak{D} \in \mathcal{D}$ is preserved by the function μ and not preserved by the function λ, and so $\operatorname{Pres}_0 \mathcal{D} = D_2^{\Uparrow}(D)$.*
2. *any set $\mathfrak{D} \in \mathcal{D}$ is preserved by the function λ and not preserved by the function μ, and so $\operatorname{Pres}_0 \mathcal{D} = L_4^{\Uparrow}(D)$.*
3. *any set $\mathfrak{D} \in \mathcal{D}$ is preserved by both the functions λ and μ, and so $\operatorname{Pres}_0 \mathcal{D} = D_1^{\Uparrow}(D)$.*

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Remarks

- The following corollary of the theorem holds.

  Let $5 \leqslant |D| < \omega$. For any local aggregation rule $f : (\mathfrak{C}_2(D))^n \to \mathfrak{C}_2(D)$ the following two condition are equivalent:

  1. $\mathrm{Inv}\, f$ contains a non-trivial symmetric subset,
  2. $\mathrm{Inv}\, f$ is symmetric, i.e., if f preserves a set $\mathfrak{D} \subseteq \mathfrak{C}_2(D)$, then it also preserves all its symmetrical images $\mathfrak{D}_\sigma$.

  This is not true for the general case of locally Boolean sets DM.

- Also, the theorem can be taken as a *reduction theorem*. Let $5 \leqslant |D| < \omega$, and let $\mathcal{D} \subseteq \mathscr{P}(\mathfrak{C}_2(D))$ be a non-trivial symmetric set. In order to check whether there is any non-dictatorship local aggregation rule that preserves every set $\mathfrak{D} \in \mathcal{D}$, it suffices to check whether this is true for the rules λ and μ. In particular, Arrow theorem can be proved in this way: for this it suffices to verify that neither rule λ nor rule μ preserves the set of rational preferences.

- Each function from clones $D_2^{\Uparrow}(D)$ and $L_4^{\Uparrow}(D)$ has a fairly large family of symmetric sets of invariants, including rational ones.

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- All functions f in the clone $D_1^{\Uparrow}(D)$ (which includes the clones $D_2^{\Uparrow}(D)$ and $L_4^{\Uparrow}(D)$) are self-dual, i.e. satisfy the condition

$$\sigma(f(x_1, x_2, \ldots, x_n)) = f(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n))$$

for all $x_1, x_2, \ldots, x_n \in D$ and $\sigma \in S_D$. Therefore, each of them can be defined using a *set of decisive coalitions* $\mathbb{C}_f \subseteq \mathscr{P}(\{1, 2, \ldots, n\})$: for all $\{a, b\} \in [D]^2$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \{a, b\}^n$,

$$f(\boldsymbol{x}) = a \Leftrightarrow \{i \in \{1, 2, \ldots, n\} : x_i = a\} \in \mathbb{C}_f.$$

A function f belongs to clone $D_2^{\Uparrow}(D)$ if and only if the set $\mathbb{C}_f$ satisfies the following conditions: for all sets $X, Y \subseteq \{1, 2, \ldots, n\}$

1. $X \in \mathbb{C}_f$ and $X \subseteq Y$ implies $Y \in \mathbb{C}_f$ (monotonicity),
2. $X \in \mathbb{C}_f$ iff $\{1, 2, \ldots, n\} \setminus X \notin \mathbb{C}_f$.

A function f belongs to clone $L_4^{\Uparrow}(D)$ if and only if the set $\mathbb{C}_f$ consists of all subsets of odd cardinality of some set $X \subseteq \{1, 2, \ldots, n\}$ of odd cardinality. Such aggregation rules are similar to the decision-making procedure using the "counting-out game". The application of this rule in practice would be very strange, however, surprisingly, it has many good properties, including the fact that it has a non-trivial symmetric invariant set of choice functions.

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1. $X \in \mathbb{C}_f$ and $X \subseteq Y$ implies $Y \in \mathbb{C}_f$ (monotonicity),
2. $X \in \mathbb{C}_f$ iff $\{1, 2, \ldots, n\} \setminus X \notin \mathbb{C}_f$.

A function f belongs to clone $L_4^{\Uparrow}(D)$ if and only if the set $\mathbb{C}_f$ consists of all subsets of odd cardinality of some set $X \subseteq \{1, 2, \ldots, n\}$ of odd cardinality. Such aggregation rules are similar to the decision-making procedure using the "counting-out game". The application of this rule in practice would be very strange, however, surprisingly, it has many good properties, including the fact that it has a non-trivial symmetric invariant set of choice functions.

THANK YOU FOR YOUR ATTENTION!