Locally integrable increasing processes with continuous compensators

Dmitriy Borzykh

NRU HSE

November 25, 2020

Introducion

This presentation is based on a published work

(Borzykh, 2018) Borzykh D. On a property of joint terminal distributions of locally integrable increasing processes and their compensators // Theory of Stochastic Processes. 2018. Vol. 23. No. 39 (2). P. 7-20).

I would like to thank my supervisor Prof. A. A. Gushchin for setting the problem and useful advices.

Our work is essentially based on A. A. Gushchin's article:

(Gushchin, 2018) A. A. Gushchin, The Joint Law of Terminal Values of a Nonnegative Submartingale and Its Compensator, Theory of Probability and Its Applications 62 (2018), no. 2, 216–235.

In (Gushchin, 2018) a class $\mathbb{W}$ of probability measures on the space $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ is introduced.

It includes all measures μ satisfying the following conditions:

1) $\int_{\mathbb{R}_+^2} (x+y)\,\mu(dx, dy) < \infty$,
2) $\int_{\mathbb{R}_+^2} x\,\mu(dx, dy) = \int_{\mathbb{R}_+^2} y\,\mu(dx, dy)$,
3) $\forall \lambda \geq 0$: $\int_{\{y \leq \lambda\}} x\,\mu(dx, dy) \leq \int_{\mathbb{R}_+^2} (y \wedge \lambda)\,\mu(dx, dy)$.

It is shown in (Gushchin, 2018) that the joint distribution of terminal values of an integrable increasing process and its compensator belongs to the class $\mathbb{W}$.

Conversely, given $\mu \in \mathbb{W}$ there is constructed an increasing integrable process such that the joint distribution of terminal values of the process and its compensator is μ and, moreover, the compensator is continuous.

Thus, if $X^\circ = (X_t^\circ)_{t\in[0;\,\infty)}$ is an integrable increasing process with a compensator $A^\circ = (A_t^\circ)_{t\in[0;\,\infty)}$, one can define on a certain stochastic basis another integrable increasing process $X^\star = (X_t^\star)_{t\in[0;\,\infty)}$ with a compensator $A^\star = (A_t^\star)_{t\in[0;\,\infty)}$, such that

$$\operatorname{Law}(X_\infty^\star,\, A_\infty^\star) = \operatorname{Law}(X_\infty^\circ,\, A_\infty^\circ)\,. \tag{1}$$

Moreover, the compensator $A^\star$ is continuous.

The main goal of the article is to extend the last statement to the locally integrable case. Namely, we state the following theorem.

Theorem (Main Theorem: Borzykh, 2018, Theorem 1.1)

For any locally integrable increasing process $X^\circ = (X^\circ_t)_{t\in[0;\,\infty)}$ *with a compensator* $A^\circ = (A^\circ_t)_{t\in[0;\,\infty)}$ *on some stochastic basis there exists another locally integrable increasing process* $X^\star = (X^\star_t)_{t\in[0;\,\infty)}$ *with a compensator* $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$*, such that relation (1) holds, as well as* $A^\star$ *is continuous.*

In our further constructions we will need the following result.

Theorem (Gushchin, 2018, Theorem 2.1)

(i) Let X be a nonnegative submartingale of class (D), $X_0 = 0$, with the Doob–Meyer decomposition $X = M + A$ into a sum of a uniformly integrable martingale M and a predictable integrable increasing process A, and let T be a stopping time. Then $\mathrm{Law}(X_T, A_T) \in \mathbb{W}$.

(ii) Let $\mu \in \mathbb{W}$. Then on some stochastic basis there exists an increasing process X with compensator A such that $\mathrm{Law}(X_\infty, A_\infty) = \mu$.

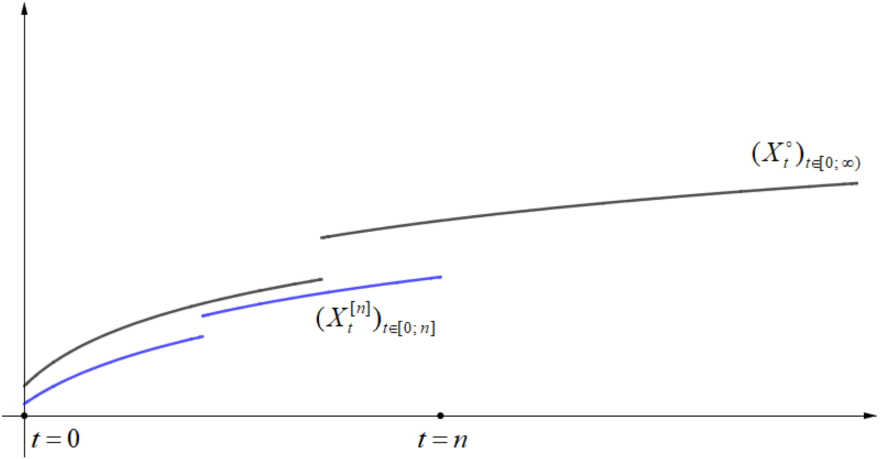
Definition

We shall call an adapted process $X = (X_t)_{t\in[0;\infty)}$ a *generalized increasing process*, if the process $X_t - X_0$, $t \in [0;\infty)$, is increasing process.

Lemma (Extension Lemma)

Let a locally integrable generalized increasing process $X^\circ = (X_t^\circ)_{t\in[0;\,\infty)}$ *such that* $\mathbb{E}[X_n^\circ] < \infty$, *for any* $n \in \mathbb{N}$, *be given on a stochastic basis* $\mathbb{B}^\circ := (\Omega^\circ, \mathcal{F}^\circ, \mathbb{P}^\circ, (\mathcal{F}_t^\circ)_{t\in[0;\,\infty)})$, *and* $A^\circ = (A_t^\circ)_{t\in[0;\,\infty)}$ *being its compensator, i.e.* A° *is predictable process with right-continuous nondecreasing trajectories such that the process* $(X_t^\circ - X_0^\circ) - (A_t^\circ - A_0^\circ)$, $t \in [0;\,\infty)$, *being a local martingale. Let also another integrable increasing process* $X^{[n]} = (X_t^{[n]})_{t\in[0;\,n]}$ *on a different stochastic basis* $\mathbb{B}^{[n]} := (\Omega^{[n]}, \mathcal{F}^{[n]}, \mathbb{P}^{[n]}, (\mathcal{F}_t^{[n]})_{t\in[0;\,n]})$, $n \in \mathbb{N}$, *with a compensator* $A^{[n]} = (A_t^{[n]})_{t\in[0;\,n]}$ *be given. Moreover,*

$$\mathrm{Law}\left(\begin{bmatrix} X_0^{[n]} \\ A_0^{[n]} \end{bmatrix}, \begin{bmatrix} X_n^{[n]} \\ A_n^{[n]} \end{bmatrix}\right) = \mathrm{Law}\left(\begin{bmatrix} X_0^\circ \\ A_0^\circ \end{bmatrix}, \begin{bmatrix} X_n^\circ \\ A_n^\circ \end{bmatrix}\right).$$



$$\mathrm{Law}\left(\begin{bmatrix} X_0^{[n]} \\ A_0^{[n]} \end{bmatrix}, \begin{bmatrix} X_n^{[n]} \\ A_n^{[n]} \end{bmatrix}\right) = \mathrm{Law}\left(\begin{bmatrix} X_0^\circ \\ A_0^\circ \end{bmatrix}, \begin{bmatrix} X_n^\circ \\ A_n^\circ \end{bmatrix}\right)$$

Lemma (Extension Lemma)

Then one can define a pair of processes $X^{[n+1]} = (X_t^{[n+1]})_{t\in[0;\,n+1]}$ *and* $A^{[n+1]} = (A_t^{[n+1]})_{t\in[0;\,n+1]}$ *on a certain extension* $\mathbb{B}^{[n+1]} := (\Omega^{[n+1]}, \mathcal{F}^{[n+1]}, \mathbb{P}^{[n+1]}, (\mathcal{F}_t^{[n+1]})_{t\in[0;\,n+1]})$ *of a stochastic basis* $(\Omega^{[n]}, \mathcal{F}^{[n]}, \mathbb{P}^{[n]}, (\mathcal{F}_t^{[n]})_{t\in[0;\,n]})$*, satisfying the following conditions:*

(i) $X^{[n+1]}$ *is an integrable increasing process, and process* $A^{[n+1]}$ *is its compensator,*

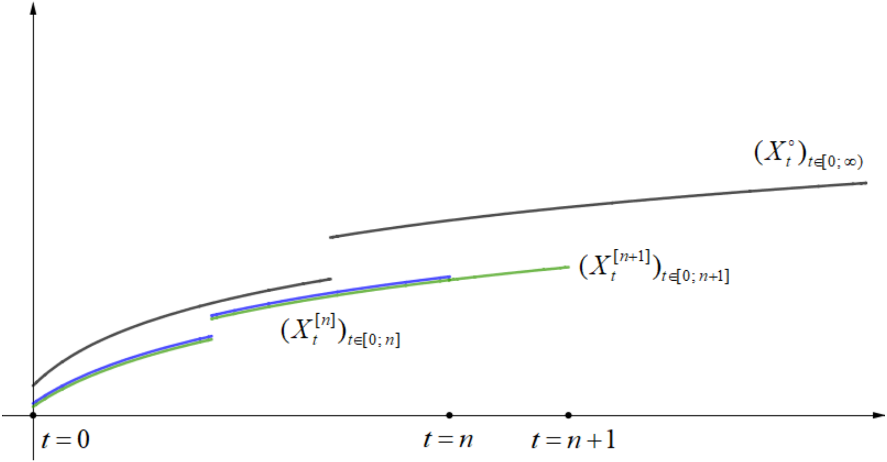
(ii) *the processes* $(X_t^{[n]})_{t\in[0;\,n]}$ *and* $(X_t^{[n+1]})_{t\in[0;\,n]}$ *coincide,*

(iii) *the processes* $(A_t^{[n]})_{t\in[0;\,n]}$ *and* $(A_t^{[n+1]})_{t\in[0;\,n]}$ *coincide,*

(iv)

$$\mathrm{Law}\left(\begin{bmatrix} X_0^{[n+1]} \\ A_0^{[n+1]} \end{bmatrix}, \begin{bmatrix} X_n^{[n+1]} \\ A_n^{[n+1]} \end{bmatrix}, \begin{bmatrix} X_{n+1}^{[n+1]} \\ A_{n+1}^{[n+1]} \end{bmatrix}\right) = \mathrm{Law}\left(\begin{bmatrix} X_0^{\circ} \\ A_0^{\circ} \end{bmatrix}, \begin{bmatrix} X_n^{\circ} \\ A_n^{\circ} \end{bmatrix}, \begin{bmatrix} X_{n+1}^{\circ} \\ A_{n+1}^{\circ} \end{bmatrix}\right),$$

(v) *process* $(A_t^{[n+1]})_{t\in[n;\,n+1]}$ *is continuous.*



$$\mathrm{Law}\left(\begin{bmatrix} X_0^{[n+1]} \\ A_0^{[n+1]} \end{bmatrix}, \begin{bmatrix} X_n^{[n+1]} \\ A_n^{[n+1]} \end{bmatrix}, \begin{bmatrix} X_{n+1}^{[n+1]} \\ A_{n+1}^{[n+1]} \end{bmatrix}\right) = \mathrm{Law}\left(\begin{bmatrix} X_0^{\circ} \\ A_0^{\circ} \end{bmatrix}, \begin{bmatrix} X_n^{\circ} \\ A_n^{\circ} \end{bmatrix}, \begin{bmatrix} X_{n+1}^{\circ} \\ A_{n+1}^{\circ} \end{bmatrix}\right)$$

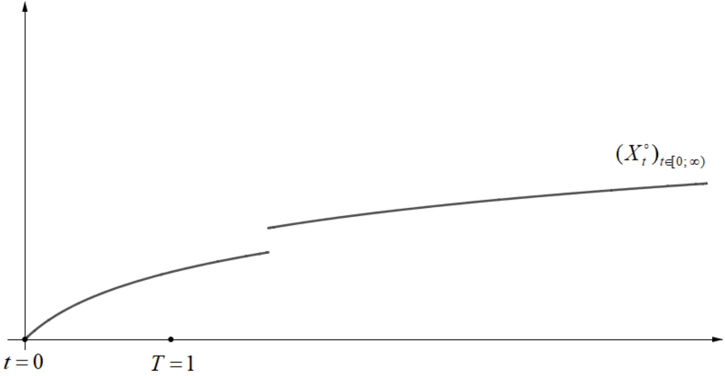
The proof of Extension Lemma goes along the same lines as the proof of Lemma 3.1 in (Borzykh, 2018).

Proof of Main Theorem

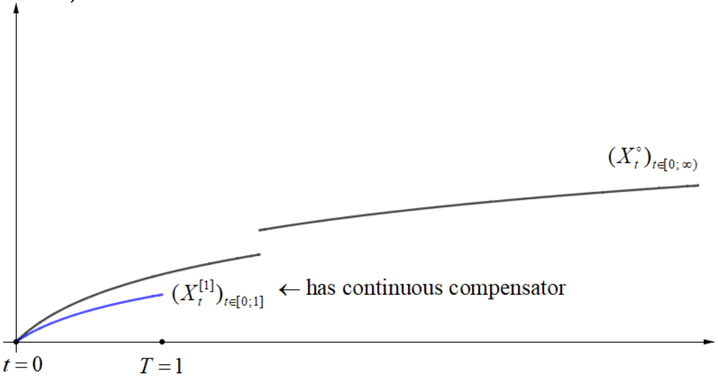
Let a locally integrable increasing process $X^\circ = (X^\circ_t)_{t\in[0;\,\infty)}$ and a localizing sequence of finite stopping times $(T_n)_{n=1}^{\infty}$ be given. It can be shown that without loss of generality one can assume that $T_n = n$, $n \in \mathbb{N}$ (for details see (Borzykh, 2018)).

We start with the following recursive procedure.

Step 1. Applying Theorem 2.1 (i) (Gushchin, 2018) to the integrable increasing process $(X_t^\circ)_{t\in[0;1]}$, as well as its compensator $(A_t^\circ)_{t\in[0;1]}$ and a stopping time $T = 1$, we get $\mathrm{Law}\,(X_1^\circ, A_1^\circ) \in \mathbb{W}$.



Then by Theorem 2.1 (ii) (Gushchin, 2018), there exists a stochastic basis $\mathbb{B}^{[1]} := \left(\Omega^{[1]}, \mathcal{F}^{[1]}, \mathbb{P}^{[1]}, (\mathcal{F}_t^{[1]})_{t\in[0;1]}\right)$, and an integrable process $(X_t^{[1]})_{t\in[0;1]}$ on it with a continuous compensator $(A_t^{[1]})_{t\in[0;1]}$, such that $\mathrm{Law}\left(X_1^{[1]}, A_1^{[1]}\right) = \mathrm{Law}\,(X_1^{\circ}, A_1^{\circ})$.



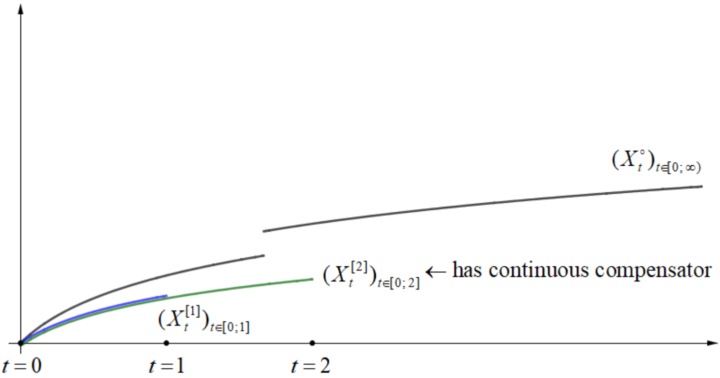
Step $n = 2$. Remark that the pair of processes $(X_t^\circ)_{t\in[0;\infty)}$ and $(A_t^\circ)_{t\in[0;\infty)}$ and the pair of processes $(X_t^{[1]})_{t\in[0;1]}$ and $(A_t^{[1]})_{t\in[0;1]}$ fit the requirements of Extension Lemma.

So, applying this lemma, we build a stochastic basis

$$\mathbb{B}^{[2]} := \left(\Omega^{[2]}, \mathcal{F}^{[2]}, \mathbb{P}^{[2]}, (\mathcal{F}_t^{[2]})_{t\in[0;2]}\right),$$

and an integrable increasing process $(X_t^{[2]})_{t\in[0;2]}$ with a continuous compensator $(A_t^{[2]})_{t\in[0;2]}$, satisfying the condition

$$\mathrm{Law}\left(X_2^{[2]}, A_2^{[2]}\right) = \mathrm{Law}\left(X_2^\circ, A_2^\circ\right).$$


$(X_t^{\circ})_{t\in[0;\infty)}$
$(X_t^{[2]})_{t\in[0;2]}$ ← has continuous compensator
$(X_t^{[1]})_{t\in[0;1]}$
$t=0$
$t=1$
$t=2$

All the steps starting from the second are performed similarly.

Step $n+1$, $n \geq 2$. Remark that the pair of processes $(X_t^{\circ})_{t\in[0;\,\infty)}$ and $(A_t^{\circ})_{t\in[0;\,\infty)}$ and the pair of processes $(X_t^{[n]})_{t\in[0;\,n]}$ and $(A_t^{[n]})_{t\in[0;\,n]}$ fit the requirements of Extension Lemma.

So, applying this lemma, we build a stochastic basis

$$\mathbb{B}^{[n+1]} := \big(\Omega^{[n+1]},\, \mathcal{F}^{[n+1]},\, \mathbb{P}^{[n+1]},\, (\mathcal{F}_t^{[n+1]})_{t\in[0;\,n+1]}\big),$$

and an integrable increasing process $(X_t^{[n+1]})_{t\in[0;\,n+1]}$ with a continuous compensator $(A_t^{[n+1]})_{t\in[0;\,n+1]}$, satisfying the condition

$$\mathrm{Law}\left(X_{n+1}^{[n+1]},\, A_{n+1}^{[n+1]}\right) = \mathrm{Law}\left(X_{n+1}^{\circ},\, A_{n+1}^{\circ}\right).$$

Now, we are ready to define the required stochastic basis

$$\mathbb{B}^\star := \big(\Omega^\star, \mathcal{F}^\star, \mathbb{P}^\star, (\mathcal{F}_t^\star)_{t\in[0;\infty)}\big)$$

and a locally integrable increasing process $X^\star = (X_t^\star)_{t\in[0;\infty)}$ on it with a continuous compensator $A^\star = (A_t^\star)_{t\in[0;\infty)}$.

We have a sequence of extensions of stochastic basises

$$\mathbb{B}^{[1]} \Subset \ldots \Subset \mathbb{B}^{[n]} \Subset \mathbb{B}^{[n+1]} \Subset \ldots$$

and a sequence of extensions of 2-dimensional processes

$$\begin{bmatrix} X_t^{[1]} \\ A_t^{[1]} \end{bmatrix}_{t \in [0;\, 1]} \Subset \ldots \Subset \begin{bmatrix} X_t^{[n]} \\ A_t^{[n]} \end{bmatrix}_{t \in [0;\, n]} \Subset \begin{bmatrix} X_t^{[n+1]} \\ A_t^{[n+1]} \end{bmatrix}_{t \in [0;\, n+1]} \Subset \ldots$$

To construct the required objects $\mathbb{B}^\star$, $X^\star$, and $A^\star$, we "glue" these stochastic basises together into stochastic basis

$$\mathbb{B}^\star := \big(\Omega^\star,\, \mathcal{F}^\star,\, \mathbb{P}^\star,\, (\mathcal{F}_t^\star)_{t \in [0;\, \infty)}\big)$$

and "glue" these processes into process

$$\begin{bmatrix} X_t^\star \\ A_t^\star \end{bmatrix}_{t \in [0;\, \infty)} .$$

Namely, put:

$$\Omega := \mathbb{R}^2_+ \times [0;\, \infty] \times [0;\, \infty] \times [0;\, 1], \qquad \mathcal{F} := \mathcal{B}(\Omega),$$

$$\Omega^\star := \Omega^{[1]} \times (\Omega)^\infty, \quad \mathcal{F}^\star := \mathcal{F}^{[1]} \otimes \bigotimes_{i=2}^{\infty} \mathcal{F},$$

$$\mathcal{F}^\star_t := \begin{cases} \mathcal{F}^{[1]}_t \otimes \{\emptyset,\, \Omega\}^\infty, & t \in [0;\, 1], \\ \mathcal{F}^{[1]}_1 \otimes \mathcal{F}_{t-1} \otimes \{\emptyset,\, \Omega\}^\infty, & t \in (1;\, 2], \\ \mathcal{F}^{[1]}_1 \otimes \left(\bigotimes_{i=2}^{n-1} \mathcal{F}_1\right) \otimes \mathcal{F}_{t-n+1} \otimes \{\emptyset,\, \Omega\}^\infty, & t \in (n-1;\, n], \quad n \geq 3. \end{cases}$$

(for more details about Ω and $\mathcal{F}$ see Lemma 2.1 in (Borzykh, 2018).

Next, in view of the Ionescu-Tulcea theorem (see e.g. (Shiryaev, Probability, 2016, vol. 1)) on the measurable space $(\Omega^\star, \mathcal{F}^\star)$ there exists a unique probability measure $\mathbb{P}^\star$, such that

$$\forall n \in \mathbb{N} \quad \forall B^{[n]} \in \mathcal{F}^{[n]}: \quad \mathbb{P}^\star\big(B^{[n]} \times (\Omega)^\infty\big) = \mathbb{P}^{[n]}\big(B^{[n]}\big).$$

Further, let $\omega^\star = \big(\omega^{[1]}, \omega_2, \ldots, \omega_n, \ldots\big) \in \Omega^\star$. Set

$$X_t^\star(\omega^\star) := \begin{cases} X_t^{[1]}\left(\omega^{[1]}\right), & t \in [0;1], \\ X_t^{[n]}\left(\omega^{[1]}, \omega_2, \ldots, \omega_n\right), & t \in (n-1;n], \quad n \geq 2, \end{cases}$$

$$A_t^\star(\omega^\star) := \begin{cases} A_t^{[1]}\left(\omega^{[1]}\right), & t \in [0;1], \\ A_t^{[n]}\left(\omega^{[1]}, \omega_2, \ldots, \omega_n\right), & t \in (n-1;n], \quad n \geq 2, \end{cases}$$

$$M_t^\star(\omega^\star) := X_t^\star(\omega^\star) - A_t^\star(\omega^\star), \quad t \geq 0.$$

It can be shown that $M^\star = (M_t^\star)_{t\in[0;\,\infty)}$ is a martingale on $\left(\Omega^\star,\, \mathcal{F}^\star,\, \mathbb{P}^\star,\, (\mathcal{F}_t^\star)_{t\in[0;\,\infty)}\right)$ (for details see (Borzykh, 2018)).

The process $A^\star = (A_t^\star)_{t\in[0;\,\infty)}$ is a predictable (by continuity) increasing process.

Finally, formula (1) is obtained from the relations

$$\lim_{n\to\infty} (X_n^\star,\, A_n^\star) = (X_\infty^\star,\, A_\infty^\star)\,, \qquad \lim_{n\to\infty} (X_n^\circ,\, A_n^\circ) = (X_\infty^\circ,\, A_\infty^\circ)\,,$$

$$\mathrm{Law}\,(X_n^\star,\, A_n^\star) = \mathrm{Law}\,(X_n^\circ,\, A_n^\circ)\,, \quad n \in \mathbb{N},$$

and the fact that almost sure convergence implies weak convergence. □

Thank you for your attention!

Possible Applications

A complete description of the class of possible distributions of a random vector $(X^\circ_\infty, A^\circ_\infty)$ is not known yet in the locally integrable case. Some steps in this direction were made by A. A. Gushchin in recent article (Gushchin, 2018 (b)), but the final answer have not been archived yet. We believe that our Main Theorem sheds extra light on this problem and can simplify its solution.

[Gushchin, 2018 (b)] A. A. Gushchin, On possible relations between an increasing process and its compensator in the non-integrable case, Russian Mathematical Surveys 73 (2018), no. 5, 928–930.